Common Differences

A-E Strand(s): Discrete Mathematics and Algebra. Sample Courses: Middle School Course 2, Middle School One-Year Advanced Course, Integrated 1, and Geometry.

Topic/Expectation

D.C.2 Mathematical reasoning

c. Explain and illustrate the role of definitions, conjectures, theorems, proofs and counterexamples in mathematical reasoning.

Other Topic/Expectation(s)

A.A.1 Variables and expressions

d. Identify and transform expressions into equivalent expressions.

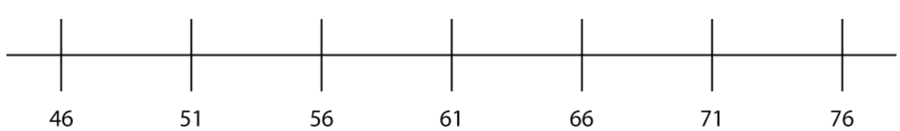
Rationale

This task illustrates mathematical reasoning using a general conjecture that students can prove by building from specific examples to a general case. The proof is incorporated into an algebraic context, something not often done in most textbooks.

Instructional Task

An arithmetic sequence is a list of real numbers in which each term is the sum of the previous term and a constant (called the common difference). For example, 46, 51, 56, 61, 66, 71, 76 is an arithmetic sequence of seven numbers with first number 46 and each subsequent number 5 more than the previous number. Five is the *common difference* of the sequence, and each of the seven numbers is a *term* of the sequence. The sum of the terms of a sequence is called a *series.*

On a number line, an arithmetic sequence is represented by a set of points that have a constant interval between each successive point.



1. Find a shortcut for finding the sum of the seven terms in this sequence. State your shortcut as a mathematical conjecture, using clear and correct mathematical language. Make a mathematically convincing argument to demonstrate that your conjecture works.

2. Test your conjecture to determine whether it works for finding the sum of an arithmetic sequence with 5 terms. What about for a series with 9 terms? Will your conjecture work for a series with 6 terms? Modify your conjecture, if necessary, so it will work in each of these situations.

3. Refine your conjecture to describe a general technique that will work for finding the sum of the terms in any arithmetic sequence, regardless of whether the sequence has an even or an odd number of terms. Make a mathematically convincing argument for your conjecture.

Discussion/Further Questions/Extensions

Depending on the students' levels of understanding, the teacher may want to use a sequence with an even number of terms to begin this task, rather than beginning with an odd number of terms. This may help some students see shortcuts more easily.

Algebra II students should be able to complete this task without much scaffolding, especially if they work in groups, an appropriate approach for this kind of task. Providing too much scaffolding can unnecessarily lower the complexity and richness of the task. However, depending on students' experience, teachers may choose to spur students' thinking by guiding them to find pairs of terms that average to the mean.

Note: As a point of reference, teachers may recall that the sum of consecutive integers can be represented as (1 + 2 + 3 + . . . + n) = $\sum_{1}^{n} n = \frac{n(n+1)}{2}$.

Extension

Depending on students' levels of understanding and their past experiences, students could be asked to prove their conjectures rather than make a convincing argument. This would either build on or stimulate discussion about what is required of an algebraic proof.

Sample Solutions

1. Find a shortcut for finding the sum of the seven terms in this sequence. State your shortcut as a mathematical conjecture, using clear and correct mathematical language. Make a mathematically convincing argument to demonstrate that your conjecture works.

   Answers may vary.

   Conjecture: *The sum of the seven terms is equal to seven multiplied by the middle term in the sequence. 46 + 51 + 56 + 61 + 66 + 71 + 76 = 427 = 7(61).*

   Possible argument for method 1 *(based on how far each term is from the middle term):*

   First rewrite all the terms based on the middle value (61) plus or minus the appropriate value that will yield the number at that place in the sequence:

   = (61 – 15) + (61 – 10) +(61 – 5) + 61 + (61+5) + (61 + 10) + (61 +15)

   Then use the commutative and associative properties of equality to rearrange and group compatible terms:

   = 61 +61 +61 +61 +61 +61 +61– 15 +15 – 10 + 10 – 5 +5

Use the additive inverse and additive identity properties of equality to simplify:

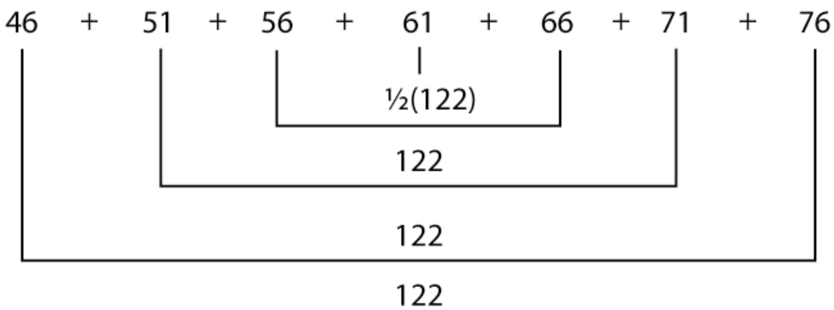
= 61 +61 +61 +61 +61 +61 +61

Rewrite the multiple addition in terms of multiplication:

= 7(61)

Possible argument for method 2 *(based on pairing the numbers, as in the Gaussian method):*

Adding the first and last terms together gives the same sum as adding the second and second to last terms together, and so on. The middle term is half of that sum:



Using the distributive property of equality, add the three sums and middle term together.

Change the mixed number to an improper fraction to simplify the calculation. This gives us the expression of half the number of terms times the sum of the first and last number.

Using the commutative and associative properties, rearrange and regroup the terms to simplify to seven times the middle term, matching the first method. This simplification is also equal to the number of terms times the mean of the first and last terms.

$$\begin{aligned} 3(122)+\frac{1}{2}(122) &= 3\frac{1}{2}(122) \\ &= \frac{7}{2}(122) \\ &= 7\left(\frac{122}{2}\right) \\ &= 7(61) \end{aligned}$$

2. Test your conjecture to determine whether it works for finding the sum of an arithmetic sequence with 5 terms. What about for a series with 9 terms? Will your conjecture work for a series with 6 terms? Modify your conjecture, if necessary, so it will work in each of these situations.

Method 1:

5 terms

46 + 51 + 56 + 61 + 66 = 5(56)

280 = 280

It works!

9 terms

$$46 + 51 + 56 + 61 + 66 + 71 + 76 + 81 + 86 = 9(66)$$
$$594 = 594$$

It works!

6 terms

$$46 + 51 + 56 + 61 + 66 + 71 = 6(?)$$

It does not work, as there is no middle term. However, it would work with the mean of the two middle terms, $\frac{(56+61)}{2} = 58.5$.

$$46 + 51 + 56 + 61 + 66 + 71 = 6(58.5)$$
$$351 = 351$$

Now it works!

Method 2:

5 terms

$$\begin{aligned} 46 + 51 + 56 + 61 + 66 &= (46 + 66) + (51 + 61) + 56 \\ &= 2(112) + \frac{1}{2}(112) \\ &= \frac{5}{2}(112) \\ 280 &= 280 \end{aligned}$$

It works!

9 terms

$$\begin{aligned} 46 + 51 + 56 + 61 + 66 + 71 + 76 + 81 + 86 &= (46 + 86) + (51 + 81) + (56 + 76) + (61 + 71) + 66 \\ &= 4(132) + \frac{1}{2}(132) \\ &= \frac{9}{2}(132) \\ 594 &= 594 \end{aligned}$$

It works!

6 terms

$$\begin{aligned} 46 + 51 + 56 + 61 + 66 + 71 &= (46 + 71) + (51 + 66) + (56 + 61) \\ &= 3(117) \\ 351 &= 351 \end{aligned}$$

It works, even though there is no middle number. In fact, having an even number of terms in the sequence eliminates fractions, so the calculation is simpler.

3. Refine your conjecture to describe a general technique that will work for finding the sum of the terms in any arithmetic sequence, regardless of whether the sequence has an even or an odd number of terms. Make a mathematically convincing argument for your conjecture.

Answers may vary. Two possible solutions are shown:

Conjecture: *The sum of the terms in an arithmetic sequence is equal to half the total number of terms multiplied by the sum of the first and last terms.*

Since two terms are added together to get one sum (see Gaussian method above), half the number of terms will determine the number of sums. Since these sums are always equal to the sum of the first and last terms, then we can use this sum. This works for an even or odd number of terms.

For the general sequence:

$$a + (a + d) + (a + 2d) + \quad . \quad . \quad . \quad + (a + (n\text{-}2)d) + (a + (n\text{-}1)d) + (a + nd)$$

$$(a + 2d) + (a + (n\text{-}2)d)$$
$$2a + nd$$

$$(a + d) + (a + (n\text{-}1)d)$$
$$2a + nd$$

$$a + (a + nd)$$
$$2a + nd$$

Your number of terms will always be one more than your number of differences, making the number of terms n + 1. Your formula would be $\left(\frac{n+1}{2}\right)(2a+nd)$*, where n is the number of differences that have been added, a is the value of the first term, and d is the difference between each term of the sequence.*

***Conjecture**: The sum of the terms in an arithmetic sequence is equal to the mean of the first and last term times the number of terms.*

$$a + (a + d) + (a + 2d) + \ldots + (a + nd) = \underbrace{a + a + \ldots + a}_{(n+1\ terms)} + \underbrace{d + 2d + 3d + \ldots + nd}_{(n\ terms)} =$$

$$= a\,(n + 1) + d\,(1 + 2 + 3 + \ldots + n)$$

$$= a\,(n + 1) + d\left(\frac{n(n+1)}{2}\right) \text{ (see Discussion section)}$$

$$= 2a\left(\frac{(n+1)}{2}\right) + d\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{2a(n+1) + d[n(n+1)]}{2}$$

$$= \frac{(n+1)(a + a + dn)}{2}$$

$$= (n+1)\left(\frac{(a + a + dn)}{2}\right)$$

$$= (n+1)\left(\frac{2a + dn}{2}\right)$$

Using the commutative, associative, and distributive properties, and finding common denominators, we were able to demonstrate mathematically how the sum of an arithmetic sequence can be found by multiplying the number of terms by the mean of the first and last term of the sequence.