Bighorn Sheep

A-E Strand(s): Algebra. Sample Courses: Integrated 3, Algebra I, and Algebra II.

Topic/Expectation

A.C.1 Elementary functions

f. Recognize and solve problems that can be modeled using exponential and power functions; interpret the solution(s) in terms of the context of the problem.

Core Algebra II EOC Content

EOC: X1. Exponential functions

a. Determine key characteristics of exponential functions and their graphs.
c. Describe the effect that changes in the parameter of an exponential function have on the shape and position of its graph.
d. Recognize, express, and solve problems that can be modeled using exponential functions, including those where logarithms provide an efficient method of solution. Interpret their solutions in terms of the context.

Rationale

In this task, students model a situation with exponential functions in order to make predictions and solve problems.

Instructional Task

Among the many species that have been endangered at one time or another is the desert bighorn sheep. The desert bighorn sheep are sensitive to human-induced problems in the environment and their numbers are therefore a good indicator of land health.

It is estimated that in the 1600s, there were about 1.75 million bighorn sheep in North America. By 1960, the bighorn sheep population in North America had dropped to about 17,000. There appears to have been a similar decline in west Texas, where wildlife biologists have data showing that in 1880, there were around 1,500 bighorn sheep in west Texas and by 1955, the population had dwindled to 25 in that area. Efforts to reintroduce desert bighorn sheep in west Texas began around 1957 and by 1993, there were about 400 desert bighorn sheep in west Texas roaming free or in captivity.

1. Assume that the annual percentage decrease in the bighorn sheep population in west Texas was fairly constant from 1880 to 1955. Model the bighorn sheep population for any year in this range with an exponential function of the form $P = ab^t$, where t is the number of years since 1880, a is the population of the bighorn sheep in west Texas in 1880, b is the annual rate of decay, and P is the annual bighorn sheep population in west Texas for the given year.

2. Assume that, starting in 1957 when reintroduction began, the annual percentage increase in the bighorn sheep population in west Texas was fairly constant. Model the bighorn sheep

population for any year since 1957 with the exponential function, $P = ab^t$, where t is the number of years since 1957, b is the annual rate of growth, P is the annual bighorn sheep population in west Texas for the given year, and a is the bighorn sheep population in 1957.

3. Describe the mathematical domain and range values for the two functions in questions 1 and 2. Describe reasonable domain and range values for the actual situations modeled by these functions. Explain any differences between the two types of domains and ranges (mathematical and situational).

4. From 1880 to 1955, by what percentage was the population decreasing annually? From 1957 to 1993, by what percentage was the population increasing annually?

5. By what year had the sheep population dropped to 750 or fewer? Use technology (tables and/or graphs) and algebraic methods to determine this. Explain your reasoning.

6. If the reintroduction program continues, in what year will the bighorn sheep population again be at least 750? Use technology (tables and/or graphs) and algebraic methods to determine this. Explain your reasoning.

7. In 2001, it was reported that there were 500 bighorn sheep in west Texas. Is this number consistent with the number predicted by the exponential model for reintroduction? Why or why not?

Discussion/Further Questions/Extensions

This task provides an opportunity to discuss the similarities and differences between a constant rate of growth from a multiplicative perspective (exponential growth or decay) compared to a constant rate of growth from an additive perspective (linear growth). The constant b in the exponential function rule $P = ab^t$ plays an important role for the function that in some ways resembles the role of the constant m in the linear function rule $y = mx + b$. Students may benefit from considering the differences in exponential functions when the value of b is greater than one versus when it is less than one.

As a topic for extension discussion, students can explore the concepts of half-life and doubling time in other types of problem situations.

Extension questions:

1. Why are exponential functions used to model decline and growth in desert bighorn sheep populations?

2. In question 5, you found that the 1880 population of 1,500 bighorn sheep had declined to 750 (one-half its size) in 12.7 years. In other words, its half-life was 12.7 years. After another 12.7 years, what will the population be?

3. Suppose wildlife experts develop a new reintroduction program with a predicted population increase of 5% to 8% per year. For this range of percentage increase, how many years might it take the population to double?

Sample Solutions

1. Assume that the annual percentage decrease in the bighorn sheep population in west Texas was fairly constant from 1880 to 1955. Model the bighorn sheep population for any year in this range with an exponential function of the form $P = ab^t$, where t is the number of years since 1880, a is the population of the bighorn sheep in west Texas in 1880, b is the annual rate of decay, and P is the annual bighorn sheep population in west Texas for the given year.

   Consider the independent variable for the situation to be time, t, in years since 1880. This means that the year 1880 would correspond to t = 0, when the sheep population is 1500.

   Likewise, 1955 would correspond to t = 75, when the sheep population is 25.

   The general model for exponential functions is $P = ab^t$. *Substituting 0 for t and 1500 for P gives a = 1500, regardless of the value of b. When written in the form (t, P(t)), these two data points are (0, 1500) and (75, 25).*

   Use the point (75, 25) to find the value of b. Substitute 75 for t and 25 for P in the function rule, and solve for b.

   $1500b^{75} = 25$

   $b^{75} = 0.0167$

   $b = (0.0167)^{1/75}$

   $b = 0.947$

   Thus, the rate of decay for this situation is 0.947. The model for the decreasing bighorn sheep population from 1880 through 1955, therefore, is $P = 1500(0.947)^t$.

2. Assume that, starting in 1957 when reintroduction began, the annual percentage increase in the bighorn sheep population in west Texas was fairly constant. Model the bighorn sheep population for any year since 1957 with the exponential function $P = ab^t$, where t is the number of years since 1957, b is the annual rate of growth, P is the annual bighorn sheep population in west Texas for the given year, and a is the bighorn sheep population in 1957.

   For the reintroduction model, let 1957 correspond to time 0 (t = 0). Then 1993 will correspond to time 36 (t = 36).

   Using the model given in question 1, we can determine the population in 1957. Since 77 years separate 1880 and 1957, t = 77.

   $P = 1500(0.947)^t$

   $= 1500(0.947)^{77}$

   $= 22.649$

   ≈ 23

Two data points that can be used for our new model might be based on the years 0 and 36, or (0, 23) and (36, 400). With t = 0 and P = 23, P = ab^t gives a = 23.

Now we can substitute t = 36 and P = 400 in $P = 23b^t$ *to solve for b.*

$$23b^{36} = 400$$

$$b^{36} = 17.39$$

$$b = 17.39^{1/36}$$

$$b \approx 1.083$$

The model for the increasing population is $P = 23(1.083)^t$.

3. Describe the mathematical domain and range values for the two functions in questions 1 and 2. Describe reasonable domain and range values for the actual situations modeled by these functions. Explain any differences between the two types of domains and ranges (mathematical and situational).

   Since these are exponential models, the mathematical domain for both models is the set of all real numbers. Since there is no vertical shift, the range is the set of all positive real numbers.

   Domain $(-\infty, \infty)$ *or* $\{x: -\infty < x < \infty\}$

   Range $(0, \infty)$ *or* $\{y: y > 0\}$

   For this particular situation, the domain for the decreasing population model is the set of real numbers from 0 to 77 inclusive (corresponding to the years from 1880 to 1955). Students might make a case for the domain consisting of integers only; however, we might be interested in growth at different points during a year, which supports a domain consisting of real numbers rather than integers. The range for the model is the set of integers from 25 to 1500, corresponding to the rounded values for the domain {25, 26, 27, . . .1500} where, in a table of values, we are rounding down to the nearest sheep in the annual count.

   The domain for the increasing population model is the set of non-negative real numbers with t = 0 corresponding to 1957. Since the reintroduction project continues, there appears to be no upper bound on the domain. The range is the set of integers beginning with 23 {23, 24, 25, 26, 27, ...}. Students should justify reasonable maximum values for both the domain and range.

   The domain and, therefore, the range will be restricted by practical concerns such as space and available food and water for the sheep, as well as reasonable limits on the years to be modeled.

4. From 1880 to 1955, by what percentage was the population decreasing annually? From 1957 to 1993, by what percent was the population increasing annually?

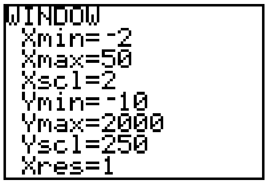
   In the decreasing population model, b = 0.947; thus, the annual percentage decrease is about 5.3% because 1 – 0.947 = 0.053.

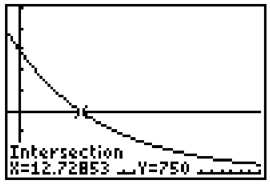
   In the increasing population model, b = 1.0825; thus, the annual percentage increase is about 8.3% because 1.083 = 1 + 0.083.

5. By what year had the sheep population dropped to 750 or fewer? Use technology (tables and/or graphs) and algebraic methods to determine this. Explain your reasoning.

   To determine how many years it takes the decreasing population to drop to at most 750 sheep, we use the calculator's graph or table functions. Let $Y_1 = 1500(0.947)^x$ *and*

   $Y_2 = 750$*. Graph the functions and find the point of intersection.*





   The graph shows that it takes nearly 13 years for the population to decrease to 750 sheep.

   The table shows that in 12 years, the population has dropped to 780 sheep, and in 13 years, it has dropped to 738.

X	Y1	Y2
9	918.84	750
10	870.14	750
11	824.03	750
12	780.35	750
13	738.99	750
14	699.83	750
15	662.74	750

Y1=738.99412556

   To see this algebraically, we must solve the equation:

$$1500(0.947)^t = 750$$

$$(0.947)^t = 0.5$$

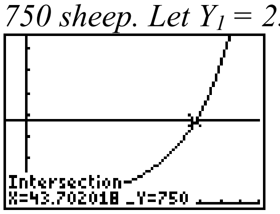
$$t = \frac{\ln(0.5)}{\ln(0.947)}$$

$$t \approx 12.729$$

   Toward the end of 1893 (since 1880 + 13 = 1893), the sheep population would be about 750. From that point on, it would get smaller, since the population is decreasing.

6. If the reintroduction program continues, in what year will the bighorn sheep population again be at least 750? Use technology (tables and/or graphs) and algebraic methods to determine this. Explain your reasoning.

   We use the same procedure to determine when the increasing population will again reach 750 sheep. Let $Y_1 = 23(1.083)^x$ *and* $Y_2 = 750$*.*



X	Y1	Y2
41	604.64	750
42	654.82	750
43	709.17	750
44	768.03	750
45	831.78	750
46	900.82	750
47	975.59	750

X=44

It will take 44 years after 1957 for the sheep population to reach 750.

To show this algebraically, we solve the equation:

$$23(1.083)^t = 750$$

$$1.083^t = 32.609$$

$$t = \frac{\ln(32.609)}{\ln(1.083)}$$

$$t \approx 43.702$$

Sometime during 2001 (since 1957 + 44 = 2001), the population should have hit 750 again and then continued to increase.

7. In 2001, it was reported that there were 500 bighorn sheep in west Texas. Given the reintroduction model, is this reasonable? Why or why not?

   The model for the increasing population shows that in 2001, there should have been about 768 bighorn sheep in west Texas. The data showed only 500. The model assumes that annual growth occurs at a constant rate. This assumption is probably not realistic for this situation because a number of factors can affect the size of the sheep population— weather, disease, predators, food and water supply, etc.