

A Graduation Present

Suppose your grandparents offer you \$3,500 as a graduation gift. However, you will receive the gift only if you agree to invest the money for at least 4 years. At that time, you hope to purchase a new car as a college graduation present to yourself and hope to make a downpayment of \$5,000.

1. At what interest rate, compounded monthly, would you need to invest your money so that you have at least \$5,000 accumulated in 4 years?
2. If you invest your money at a 5% interest rate compounded daily, how long would it take you to accumulate \$5,000?
3. You decide to ask your grandparents for more money so you can plan to make a larger downpayment. You plan to save this money at a credit union that offers 6% interest compounded continuously. How much should you ask for if you want at least \$5,000 in the account in 4 years?

You know your grandfather will see the need for more money more clearly if you present the information in a table or graph. However, your grandmother, a retired math teacher, will expect an algebraic explanation.



Notes

Materials:

Graphing calculator

Note: Prerequisite experiences for this problem are that students will have been exposed to the interest formulas.

Compound interest:

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ where n is the number of times compounded each year, P is the amount invested, A is the amount in the account after t years, and r is the annual interest rate.

Continuous interest:

$A = Pe^{rt}$ where P is the amount invested, A is the amount in the account after t years, and r is the annual interest rate.

Scaffolding Questions:

- If interest is computed n times per year, what would be the function rule expressing the amount of money (A dollars) in a savings account as a function of the number of years (t) the money is in the account?
- What function rule do you use if interest is computed continuously?
- As you solve for different parameters in these functions, what types of equations do you encounter?
- What methods do you have for solving exponential equations?
- What methods seem to work best in these situations?

Sample Solutions:

1. The function expressing the amount of money, A dollars, in an account in terms of years of deposit, t years, is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is your initial deposit, r is the annual interest rate, and n is the number of compounding periods per year.

Since $P = 3,500$, $n = 12$, $t = 4$, and $A = 5,000$ we have

$$\begin{aligned} 3,500\left(1 + \frac{r}{12}\right)^{12 \cdot 4} &\geq 5,000 \\ \left(1 + \frac{r}{12}\right)^{48} &\geq \frac{5,000}{3,500} \\ 1 + \frac{r}{12} &\geq \left(\frac{10}{7}\right)^{\left(\frac{1}{48}\right)} \\ &= 1.007458 \\ \frac{r}{12} &\geq 0.007458 \\ r &\geq 0.0895 \end{aligned}$$

Therefore, we would need to find a savings program that paid at least 8.95% interest compounded monthly.

2. In this case, we know $P = 3,500$, $r = 0.05$, $n = 365$, and $A = 5,000$. We must solve for t as follows:

$$3,500\left(1 + \frac{0.05}{365}\right)^{365t} = 5,000$$

$$(1.000136986)^{365t} = \frac{10}{7}$$

Take the natural log of both sides and apply the power and quotient properties of logarithms to get

$$365t(\ln(1.000136986)) = \ln(10) - \ln(7)$$

$$365t = \frac{0.3566749}{0.0001369766}$$

$$= 2,603.905424$$

$$t = 7.134$$

Therefore, it would require more than seven years to accumulate \$5,000.

3. Now we need to determine how much to deposit initially if interest is compounded continuously, $r = 0.06$, $t = 4$, $A = 5,000$.

Use the continuous compound interest function

$$P(e^{0.06 \cdot 4}) = 5,000$$

$$P(1.1271249) = 5,000$$

$$P = 3,933.139$$

We would need \$3,933.14 as our initial deposit.

$$\$3,933.14 - \$3,500 = \$433.14$$

Algebra II TEKS Focus:
(2A.11) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(D) determine solutions of exponential and logarithmic equations using graphs, tables, and algebraic methods.

(E) determine solutions of exponential and logarithmic inequalities using graphs and tables.

(F) analyze a situation modeled by an exponential function, formulate an equation or inequality, and solve the problem.

- Show that this is true for any amount invested at a rate, r , compounded n times per year for t years.

The original amount invested is represented by P . The amount earned is

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$

The interest earned is the amount earned minus the amount invested.

$$P \left(1 + \frac{r}{n} \right)^{nt} - P$$

If the amount is multiplied by 2, it is $2P$. The amount earned is

$$2P \left(1 + \frac{r}{n} \right)^{nt}$$

The interest earned is the amount earned minus the amount invested.

$$2P \left(1 + \frac{r}{n} \right)^{nt} - 2P = 2 \left\{ P \left(1 + \frac{r}{n} \right)^{nt} - P \right\}$$

If the amount invested is multiplied by two, then the amount earned is always twice the original amount earned.

The student work on the next page demonstrates use of algebraic process and skills, but does not communicate the solution strategy.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

A Graduation Present

$$Y = A(1 + R/12)^{X \cdot 12} \text{ (monthly)}$$

R = interest rate

X = # of times compounded per year

$$\textcircled{1} * 5000 = 3500 \left(1 + \frac{R}{12}\right)^{\uparrow (4)(12)} \uparrow (X)$$

$$* \frac{5000}{3500} = \frac{3500}{3500} \left(1 + \frac{R}{12}\right)^{(48)}$$

$$* 1.428^{(48)} = \left(1 + \frac{R}{12}\right)^{48}$$

$$* 1.0074 = 1 + \frac{R}{12}$$

$$* 12 \times .0074 = R/12 \times 12$$

$$08.94 = R$$

$$\boxed{8.94\% = R}$$

This is the interest rate in which he would need to make \$5000! in 4 yr.

$$\textcircled{2} * 5000 = \frac{3500}{3500} \left(1 + \frac{.05}{365}\right)^{(X)(365)} \text{ (Daily)}$$

$$* 1.428 = \left(1 + .0001369\right)^{365X}$$

$$1.428 = \left(1.0001369\right)^{365X}$$

$$* \frac{\log 1.428}{\log 1.0001369} = \frac{365 \times \log 1.0001369}{\log 1.0001369}$$

$$* \frac{2602.624}{365} = \frac{365X}{365}$$

$$* \boxed{7.1304 \text{ yrs} = X}$$

This is how long it will take for him to make \$5000!

$$\textcircled{3} y = Pe^{rt}$$

$$* 5000 = Pe^{.06(4)}$$

$$* 5000 = Pe^{.24}$$

$$* \frac{5000}{1.271} = \frac{P(1.271)}{1.271}$$

$$\boxed{3933.14 = P}$$

$$\begin{array}{r} 3933.14 \\ - 3500.00 \\ \hline \$ 433.13 \end{array}$$

That's how much extra we need to ask for.